# Compact Operators, Fredholm Operators and Index Theory

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***Abstract*** **– In the present master thesis two important categories of operators from Functional Analysis are presented, compact operators and Fredholm operators. These operators play a significant role in the solvability of integral and more generally linear equations. The basic properties of these operators are studied, when they are defined in separable Hilbert spaces, as well as their interconnection. Extra emphasis is given to the key characteristic of a Fredholm operator, its index. The index of such an operator determines under which conditions and in which way the corresponding linear equations can be solved, using the well-known Fredholm Alternative Theorem.**

**Keywords: Compact Operators, Fredholm Operators, Index of an Operator, Fredholm Alternative, Regularization of an Operator**

## I. INTRODUCTION

Operator Theory is a branch of Functional Analysis which developed significantly in the 20th century. Two of its most important operator classes which were extensively studied, are the compact operators and the Fredholm operators. The first indirect results on compact operators can be found in the studies of Volterra and Fredholm on Integral Equations, while the studies of Noether on some classes of singular operators led to the Fredholm operators (Douglas, 1972). The main principle which connected these operators and which makes them so important, is the Fredholm Alternative Theorem. This theorem can provide information on the solvability of a certain category of linear equations.

This study was focused on the central concept of the index of a Fredholm operator. This index is the link between Functional Analysis and Algebraic Topology and led to one of the highlights of modern Global Analysis, the famous Atiyah-Singer theorem (Dieudonne, 1985). Nevertheless, throughout this work the Functional Analysis point of view was preferred. This was due to our objective which was mainly the research of the conditions concerning the index of the operator of a linear equation, under which this equation is solved directly or needs to be transformed in an appropriate form before being solved.

The rule used for the linear equation solution is the so-called Fredholm Alternative Theorem which is widely applied in the Theory of Integral Equations. This theorem can be extended to cases where the equations have Fredholm operators of a more general kind. However, in order to extend its applicability, the Atkinson Theorem which allows the "regularization" of some singular equations, was used. The transformed equations had thus a form compatible with the Fredholm Alternative Theorem and the same solutions with the initial ones.

## II. METHODOLOGY

### A. *Compact operators*

The class of compact operators, resulted directly from the study of the integral equations. Indeed, the integral operators are the most classical examples of compact operators. The main characteristic of these operators, is that they show similar behavior with the operators in finite dimensional spaces and thus they can be easily analyzed. Let us now begin with the definition of a compact operator, before proceeding to its properties.

***Definition 1:*** *A bounded linear operator $K$ in a Hilbert space $\mathscr{H}$ ($K$:$\mathscr{H}\rightarrow\mathscr{H}$) is called* ***compact****, if it maps the closed unit ball of $\mathscr{H}$ to a relatively compact subset of $\mathscr{H}$.*

More generally, we can say that an operator is compact, if it maps bounded sets of $\mathscr{H}$ to relatively compact sets. One of the most characteristic examples of compact operators is the integral operator $K$, where:

with $k \in L_2([a,b]\times[a,b])$[1]. This operator plays a central role in the so-called Fredholm integral equations of the 2nd kind and it was actually this operator that triggered the research on the compact operators and later on the Fredholm operators. The basic properties of the compact operators are the following:

<u>*Properties:*</u>

[1] In $L_2[a,b]$ spaces, every integral operator is compact (Helemskii, 2006).

(i) If an operator $K$ is compact, his adjoint operator $K^*$ will be also compact.
(ii) Every linear combination of compact operators is a compact operator.
(iii) Every composition of bounded linear operators one of which is compact, is a compact operator.
(iv) Every converging sequence of compact operators converges to a compact operator.

From the aforementioned we can conclude that the space of the compact operators is a Banach space.

Compact operators, as a generalization of operators in finite dimensional spaces, show a relatively simple structure and preserve the images of the sets upon which they act, “small”. When we have to deal with entities like these, it is often interesting to study also entities showing the opposite behavior. Fredholm operators can be regarded as a kind of anti-compact” operators.

## B. *Fredholm operators*

Fredholm operators, as previously mentioned, resulted from the study of the compact operators in integral equations. Since their main characteristics are of “geometric” nature, we will start by defining them based on these characteristics and we will see later on in which way they are connected and result from compact operators, as somewhat “anti-compact” operators.

***Definition 2*** *(Helemskii, 2006)**:** A bounded linear operator* $T:\mathscr{H}\rightarrow\mathscr{H}$*, in a Hilbert space* $\mathscr{H}$*, is called* ***Fredholm operator*** *when:*
*(i) the dimension of its kernel* $\dim(\ker(T))$ *is finite and*
*(ii) the co-dimension of its image* $\operatorname{codim}(\mathfrak{I}(T))$ *is also finite.*

Apart from these two conditions, there is often a third one demanded so that an operator is a Fredholm one. This concerns the closedness of the image of the operator $\mathfrak{I}(T)$. However, this condition becomes superfluous, since the closedness of $\mathfrak{I}(T)$ is guaranteed from the Open Map Theorem and the condition *(ii)* of the definition.

Every operator in a finite dimensional space is obviously Fredholm. An example of a Fredholm operator in infinite dimensional Hilbert spaces, is the right shift operator $S_r$:

since $\ker(S_r) = \{\mathbf{0}\}$ with $\dim(\ker(S_r)) = 0$ and its closed image:

has $\operatorname{codim}(\mathfrak{I}(S_r)) = 1$.

The “physical”-geometrical meaning of the main conditions *(i)* and *(ii)* of a Fredholm operator is firstly that the finite dimension of the kernel shows us how “far” is the operator from being one-one and secondly, the finite co-dimension shows us how “far” is from being onto.

<u>*Properties:*</u>

(i) If operator $T$ is Fredholm, then his adjoint operator $T^*$ will be Fredholm also, with $\dim(\ker(T^*)) = \operatorname{codim}(\mathfrak{I}(T))$.
(ii) The composition of two Fredholm operators is a Fredholm operator.
(iii) The linear combination of Fredholm operators is not always a Fredholm operator.

A special category of Fredholm operators, which plays an important role in the Theory of Linear Integral Equations is the one described below:

***Proposition 1:*** *If* $K:\mathscr{H}\rightarrow\mathscr{H}$ *is a compact operator, then the operator* $I-\lambda K$ *(I the identity operator) is Fredholm for every* $\lambda\in\Re$*.*

Fredholm operators in general, are the direct generalization of the $I-\lambda K$ operators, with $K$ compact. In addition, this property indicates the fact that Fredholm operators in infinite dimensional spaces are the “furthest” operators from the compact ones. This is proven by the Atkinson theorem which we will see later on. Furthermore, since the identity operator is compact only in finite dimensional spaces, a Fredholm operator can be compact also only in spaces like these. But let us now see the very important Atkinson theorem.

***Theorem*** (*Atkinson*[2])***:*** *If* $T:\mathscr{H}\rightarrow\mathscr{H}$ *is a bounded operator in a Hilbert space* $\mathscr{H}$*, then the following are equivalent:*
*(i) T is a Fredholm operator,*
*(ii) There are compact operators* $K_1, K_2:\mathscr{H}\rightarrow\mathscr{H}$ *and bounded operators* $S_1, S_2:\mathscr{H}\rightarrow\mathscr{H}$*, such as:*

This theorem can be formulated more shortly if we say that Fredholm operators are the invertible modulo compact operators. The latter is often adopted as the definition of Fredholm operators (see i.e. Douglas, 1972). Atkinson’s theorem serves also in “detecting” Fredholm operators, something not at all easy to be done with the classical definition (Schechter, 2002).

## C. *Index of a Fredholm operator*

We saw in the previous section that Fredholm operators have some very useful geometrical properties, concerning their images and their kernels. Despite the usefulness of these properties, there is another characteristic of Fredholm operators which is the most important and the one which led to the definition and study of the special category of these operators. This characteristic is the so-called index of a Fredholm operator[3] and it is defined as follows:

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[2] The theorem contains also the corresponding proposition where instead of compact operators we have finite-rank operators (operators with finite dimensional images), operators of which the compact are the generalization in infinite dimensional spaces (see Abramovich & Aliprantis, 2002).

[3] Fredholm operators are also called operators of finite index, because of their index property.

***Definition 3:*** *Let $T:\mathscr{H}\to\mathscr{H}$ be a Fredholm operator in a Hilbert space, then the* ***index*** *of this operator is defined as the finite*[4] *integer:*

*or equivalently (Douglas, 1972):*

E.g. for the right shift operator $S_r$ we saw previously, it is:

***Proposition 2:*** *If $T:\mathscr{H}\to\mathscr{H}$ is a Fredholm operator and $K:\mathscr{H}\to\mathscr{H}$ is a compact one, then $T+K$ is a Fredholm operator with:*

This proposition shows actually that the index of a Fredholm operator $T$ remains invariant under compact perturbations, to wit, under perturbations of the form $T+K$ with $K$ compact. This is the most important Fredholm operators' property and it justifies the special study of the index, despite the fact that the latter seems less "geometrical" than the properties-conditions of Definition 2. We will close this section with a proposition which extends the conclusions extracted concerning the Fredholm operators of the form $I-\lambda K$.

***Proposition 3:*** *An operator $T:\mathscr{H}\to\mathscr{H}$ will be Fredholm with index 0 if and only if $T=T_o+K$, where $T_o$ is an invertible bounded operator and $K$ a compact one.*

## III. SELECTED RESULTS

A. *Fredholm Alternative*

We will provide here the Fredholm Alternative Theorem (FAT), one of the most important theorems for the solution of linear equations. It resulted from the study of the Fredholm integral equations of the 2nd kind. In this section it will formulated in terms of compact operators in general and not in terms of integral ones.

***Theorem*** (*FAT-1*)***:*** *Let $K:\mathscr{H}\to\mathscr{H}$ be a compact operator in the Hilbert space $\mathscr{H}$, $\lambda\in\Re^*$* and the nonhomogeneous equations:

*with corresponding homogeneous:*

*Then, one of the following is valid:*

*(i) The equations (1) and (2) have unique solutions, for every $g,j\in\mathscr{H}$ and the homogeneous ones (3) and (4) have only the zero solution.*

*(ii) The equations (3) and (4) have the same number of linearly independent solutions $\{f_i\}_{i=1,\ldots,n}$, $\{h_i\}_{i=1,\ldots,n}$ and the (1),(2) have solutions if and only if $g\perp h_i$ και $j\perp f_i$, for every $i=1,2,\ldots,n$.*

FAT refers to equations that are called ***equations of the 2nd kind with a compact operator***, as a generalization of the Fredholm integral equations of the 2nd kind. If instead of the linear equation formalism we use Operator Theory terminology, the FAT is written shortly as:

***Theorem*** (*FAT-2*)***:*** *If $K:\mathscr{H}\to\mathscr{H}$ is a compact operator in a Hilbert space and $\lambda\in\Re^*$, then:*

Taking into account what was presented in the previous section, we can say that FAT is equivalent with the proposition which states that all operators of the form $I-\lambda K$ with $K$ compact, are Fredholm with index 0.

The question which immediately comes to mind after these is whether all operators with index 0 satisfy the FAT or not. The answer is positive and it is furthermore proven that all linear equations with Fredholm operators of any index value, can be transformed so that they satisfy the FAT. As a first step, we will restate the FAT using Fredholm operator terminology this time:

***Theorem*** (*FAT-3*)***:*** *If $T:\mathscr{H}\to\mathscr{H}$ is a zero index Fredholm operator, the following apply:*

*(i) $T$ (as well as its adjoint $T^*$) will be one-to-one if and only if it is onto:*

*(ii) If* dim(ker ($T$))$\neq 0$*, then it is:*

dim(ker ($T$)) = dim(ker ($T^*$))

In cases of linear equations of the form $Tf = g$ with $T$ Fredholm with a nonzero index, the FAT does not apply directly. The equations have to be firstly "regularized", in order to obtain a form complying with the conditions of the theorem. In saying so, to be transformed to equations of the 2nd kind with a compact operator (Kress, 1999). The way the transformation is done, depends on the index of the operator each time and it is performed using the Atkinson theorem and Proposition 3. There are three distinct cases, corresponding to the cases where the index of $T$ is 0, positive or negative.

- If ind($T$) = 0, then from Proposition 3 $T$ can be written as a sum of an invertible operator $T_o$ and a compact one $K$. The linear equations subsequently becomes:

  where $K'$ is a compact operator as a composition of operators where at least one of them is

[4] The index of a Fredholm operator is always finite, since both dim(ker(·)) and codim($\Im$(·)) are finite (see Definition 2).

compact. The final equation complies with the FAT, since the operator on the left-hand side has the desired form $I-\lambda K$.

- If $\mathrm{ind}(T) > 0$, from the Atkinson theorem there will be a bounded operator $S_1: \mathscr{Y} \to \mathscr{Y}$ *(which will be called the* ***equivalent***[5] ***left regularizer*** *of T) so that:*

- If $\mathrm{ind}(T) < 0$, from the Atkinson theorem there will be a bounded operator $S_2: \mathscr{Y} \to \mathscr{Y}$ *(which will be called the* ***equivalent right regularizer*** *of T) so that:*

Since $K_1$ and $K_2$ are compact (Atkinson theorem), the regularized equations have the form $(I\text{-}K)f = g$ and consequently the FAT can be applied. Concluding the aforementioned, we can say that all linear equations of the form $Tf = g$ with $T$ a Fredholm operator, can be always solved using the Fredholm Alternative whatever their indices values are. Equations like these are often met in the Theory of Singular Integral Equations.

## IV. Conclusions

In the present work we presented two important classes of operator from Functional Analysis, the compact operators and the Fredholm ones. These operators, which in infinite dimensional spaces are found the "furthest" the former from the latter, assist in easily solving linear equations of the form $Tf = g$. The central role in this solution plays the concept of the index of a Fredholm operator. Whenever $T$ is a Fredholm operator with zero index, it can be decomposed directly as a sum of an invertible and a compact operator, so that the equation could be subsequently solved via the Fredholm Alternative. If the index of $T$ is nonzero, then there are special operators-regularizers of $T$ which transform the linear equation in a form which can be again solved with the Fredholm Alternative. The latter equations, despite their different form, have the same solutions with the initial ones.

## References


Douglas R.K. (1972). *Banach Algebra Techniques in Operator Theory*. Academic Press.

Dieudonne J.A. (1985). The Index of Operators in Banach Spaces. *Integral Equations and Operator Theory*, *8*, 580-589.

Kress R. (1999). *Linear Integral Equations*. Springer.

Helemskii, A.Ya. (2006). *Lectures and Exercises on Functional Analysis*. American Mathematical Society.

Abramovich Y.A., Aliprantis C.D. (2002). *An Invitation to Operator Theory*. American Mathematical Society.

Schechter M. (2002). *Principles of Functional Analysis.* American Mathematical Society.


[5] Equivalent because the initial and the regularized equation have the same solutions (Kress, 1999).